

# The values of the high order Bernoulli polynomials at integers and the $r$ -Stirling numbers

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**Abstract.** In this paper, we exploit the  $r$ -Stirling numbers of both kinds in order to give explicit formulae for the values of the high order Bernoulli numbers and polynomials of both kinds at integers. We give also some identities linked the  $r$ -Stirling numbers and binomial coefficients.

**Keywords.** The  $r$ -Stirling numbers; the high order Bernoulli polynomials; binomial coefficients.

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## 1 Introduction

The study of the higher order Bernoulli polynomials of the both kinds have extensively used in various branches of mathematics and have extended in various directions. For details on the higher order Bernoulli polynomials of the first kind  $B_n^{(\alpha)}(x)$  one can see [13, 14, 15, 17, 19], and, for the higher order Bernoulli polynomials of the second kind  $b_n^{(\alpha)}(x)$  one can see [1, 3, 11, 12]. These polynomials are defined by the generating function to be

$$\sum_{n \geq 0} B_n^{(\alpha)}(x) \frac{t^n}{n!} = \left( \frac{t}{\exp(t) - 1} \right)^\alpha \exp(xt), \quad (1)$$

$$\sum_{n \geq 0} b_n^{(\alpha)}(x) \frac{t^n}{n!} = \left( \frac{t}{\ln(1+t)} \right)^\alpha (1+t)^x. \quad (2)$$

The numbers  $B_n^{(\alpha)} := B_n^{(\alpha)}(0)$  are the high order Bernoulli numbers of the first kind and  $B_n := B_n^{(1)}(0)$  are the Bernoulli numbers of the second kind. Also, the numbers  $b_n^{(\alpha)} := b_n^{(\alpha)}(0)$  are the high order Bernoulli numbers of the second kind and  $b_n := b_n^{(1)}(0)$  are the Bernoulli numbers of the second kind. These numbers and polynomials are connected to the  $r$ -Stirling numbers of the first and second kind  $[n]_r$  and  $\{n\}_r$  introduced by Broder [2, 8, 9].

Recall that the number  $[n]_r$  counts the number of permutations of the set  $[n] := \{1, \dots, n\}$  into  $k$  cycles such that the elements of the set  $[r]$  are in different cycles, and, the number  $\{n\}_r$  counts the number of partitions of the set  $[n]$  into  $k$  non-empty subsets such that the elements of the set  $[r]$  are in different subsets. These numbers are determined by the generating function to be

$$\sum_{n \geq k} \left[ \begin{matrix} n+r \\ k+r \end{matrix} \right]_r \frac{t^n}{n!} = \frac{1}{k!} \frac{(-\ln(1-t))^k}{(1-t)^r}, \quad (3)$$

$$\sum_{n \geq k} \left\{ \begin{matrix} n+r \\ k+r \end{matrix} \right\}_r \frac{t^n}{n!} = \frac{1}{k!} (\exp(t) - 1)^k \exp(rt). \quad (4)$$

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By combining (1) and (4) we obtain

$$B_n^{(-k)}(r) = \binom{n+k}{k}^{-1} \left\{ \begin{matrix} n+r+k \\ k+r \end{matrix} \right\}_r, \quad r, k \in \mathbb{N}, \quad (5)$$

and by combining (2) and (3) we get

$$b_n^{(-k)}(-r) = (-1)^n \binom{n+k}{k}^{-1} \left[ \begin{matrix} n+r+k \\ k+r \end{matrix} \right]_r, \quad r, k \in \mathbb{N}. \quad (6)$$

In this paper, we give formulas for the values of the high order Bernoulli polynomials at integers in terms of the  $r$ -Stirling numbers of both kinds. In particular, we may prove that the Bernoulli numbers of the first kind admit the following representations

$$\begin{aligned} b_n &= (n+1) \binom{2n}{n}^{-1} \sum_{j=0}^n \frac{(-1)^{n+j}}{j+1} \binom{2n}{n+j} \left[ \begin{matrix} n+j \\ j \end{matrix} \right] \\ &= n \sum_{j=0}^n \frac{(-1)^j}{n+j} \binom{2n}{n+j} \left\{ \begin{matrix} n+j+1 \\ j+1 \end{matrix} \right\} \end{aligned}$$

and the Bernoulli numbers of the second kind admit the following representations

$$\begin{aligned} B_n &= n \sum_{j=0}^n \frac{(-1)^{n+j}}{n+j} \binom{2n}{n+j} \left[ \begin{matrix} n+j+1 \\ j+1 \end{matrix} \right] \\ &= (n+1) \binom{2n}{n}^{-1} \sum_{j=0}^n \frac{(-1)^j}{j+1} \binom{2n}{n+j} \left\{ \begin{matrix} n+j \\ j \end{matrix} \right\}. \end{aligned}$$

As consequences, we give in the third section some identities linked  $r$ -Stirling numbers and binomial coefficients. The mathematical tools used are the identities (5), (6) and the Melzak's formula [6, 7] given by

$$f_n(\alpha + x) = \alpha \binom{\alpha+p}{p} \sum_{j=0}^p \frac{(-1)^j}{\alpha+j} \binom{p}{j} f_n(-j+x), \quad (7)$$

where  $f$  is a polynomial of degree  $n \leq p$ ,  $\binom{x}{k} := \frac{x(x-1)\cdots(x-k+1)}{k!}$ ,  $k \geq 1$ , and  $\binom{x}{0} := 1$ .

We use also the notation

$$x^{\underline{n}} = x(x-1)\cdots(x-n+1), \quad n \geq 1, \quad x^{\overline{0}} = 1 \quad \text{and} \quad x^{\overline{n}} = x(x+1)\cdots(x+n-1), \quad n \geq 1, \quad x^{\overline{0}} = 1.$$

## 2 The values of the high order Bernoulli polynomials at integers

For such applications of (7), we consider the Bernoulli polynomials of both kinds. Indeed, the definitions (1) and (2) show that  $B_n^{(\alpha)}(0)$  and  $b_n^{(\alpha)}(0)$  represent (potential) polynomials in  $\alpha$  of degree  $\leq n$ , see [4, Thm. B, p. 141]. So, the polynomials  $b_n^{(\alpha)}(x)$  and  $B_n^{(\alpha)}(x)$  are also polynomials in  $\alpha$  of degree  $\leq n$ . This help to give new formulas for the high order Bernoulli polynomials in terms of the  $r$ -Stirling numbers. The following proposition gives formulas for the values of the high order Bernoulli polynomials of both kinds at non-positive integers in terms of the  $r$ -Stirling numbers of the first kind.

**Proposition 1** Let  $\alpha$  be a real number and  $p, q, r, n$  be non-negative integers with  $p \geq n$ . We have

$$b_n^{(\alpha)}(-r) = (\alpha + q) \binom{\alpha + p + q}{p} \sum_{j=0}^p \frac{(-1)^{n+j}}{\alpha + q + j} \binom{p}{j} \frac{[n+r+j+q]_r}{\binom{n+j+q}{n}},$$

$$B_n^{(\alpha)}(-r) = (n+1-\alpha+q) \binom{n+1-\alpha+p+q}{p} \sum_{j=0}^p \frac{(-1)^{n+j}}{n+1-\alpha+q+j} \binom{p}{j} \frac{[n+r+j+q+1]_{r+1}}{\binom{n+j+q}{n}}.$$

**Proof.** Setting  $x = -q$  and replace  $\alpha$  by  $\alpha + q$  in (7) to get

$$f(\alpha) = (\alpha + q) \binom{\alpha + q + p}{p} \sum_{j=0}^p (-1)^j \binom{p}{j} \frac{f(-j-q)}{\alpha + q + j}. \quad (8)$$

By setting  $f(x) = b_n^{(x)}(-r)$  in (8) we obtain

$$b_n^{(\alpha)}(-r) = (\alpha + q) \binom{\alpha + p + q}{p} \sum_{j=0}^p (-1)^j \binom{p}{j} \frac{b_n^{(-j-q)}(-r)}{\alpha + q + j}.$$

On using (6), the last identity being the first identity of the proposition.

Upon using the Carlitz's identity  $B_n^{(\alpha)}(x) = b_n^{(n+1-\alpha)}(x-1)$ , see [3, Eqs. (2.11), (2.12)], the second identity is equivalent to the first one.  $\square$

For  $p = n$  and  $q = 0$  in Proposition 1, we get the following corollary.

**Corollary 2** Let  $\alpha$  be a real number and  $r, n$  be non-negative integers. We have

$$b_n^{(\alpha)}(-r) = \alpha \binom{\alpha + n}{n} \binom{2n}{n}^{-1} \sum_{j=0}^n \frac{(-1)^{n+j}}{\alpha + j} \binom{2n}{n+j} \left[ \begin{matrix} n+r+j \\ r+j \end{matrix} \right]_r,$$

$$B_n^{(\alpha)}(-r) = (n+1-\alpha) \binom{2n-\alpha+1}{n} \binom{2n}{n}^{-1} \sum_{j=0}^n \frac{(-1)^{n+j}}{n+1-\alpha+j} \binom{2n}{n+j} \left[ \begin{matrix} n+r+j+1 \\ r+j+1 \end{matrix} \right]_{r+1}.$$

In particular, for  $\alpha = 1$  the values of the classical Bernoulli polynomials at non-positive integers are

$$b_n(-r) : = b_n^{(1)}(-r) = (n+1) \binom{2n}{n}^{-1} \sum_{j=0}^n \frac{(-1)^{n+j}}{j+1} \binom{2n}{n+j} \left[ \begin{matrix} n+r+j \\ r+j \end{matrix} \right]_r,$$

$$B_n(-r) : = B_n^{(1)}(-r) = n \sum_{j=0}^n \frac{(-1)^{n+j}}{n+j} \binom{2n}{n+j} \left[ \begin{matrix} n+r+j+1 \\ r+j+1 \end{matrix} \right]_{r+1}.$$

These representations show that the classical Bernoulli numbers admit the representations

$$b_n = (n+1) \binom{2n}{n}^{-1} \sum_{j=0}^n \frac{(-1)^{n+j}}{j+1} \binom{2n}{n+j} \left[ \begin{matrix} n+j \\ j \end{matrix} \right],$$

$$B_n = n \sum_{j=0}^n \frac{(-1)^{n+j}}{n+j} \binom{2n}{n+j} \left[ \begin{matrix} n+j+1 \\ j+1 \end{matrix} \right].$$

Similarly to Proposition 1, the following proposition gives formulas for the values of the high order Bernoulli polynomials of both kinds at non-negative integers in terms of the  $r$ -Stirling numbers of the second kind.

**Proposition 3** *Let  $\alpha$  be a real number and  $p, q, r, n$  be non-negative integers with  $p \geq n$ . We have*

$$B_n^{(\alpha)}(r) = (\alpha + q) \binom{\alpha + q + p}{p} \sum_{j=0}^p \frac{(-1)^j}{\alpha + q + j} \binom{p}{j} \frac{\left\{ \begin{smallmatrix} n+r+q+j \\ r+q+j \end{smallmatrix} \right\}_r}{\binom{n+q+j}{n}},$$

$$b_n^{(\alpha)}(r) = (n + 1 - \alpha + q) \binom{n + 1 - \alpha + q + p}{p} \sum_{j=0}^p \frac{(-1)^j}{n + 1 - \alpha + q + j} \binom{p}{j} \frac{\left\{ \begin{smallmatrix} n+r+q+j+1 \\ r+q+j+1 \end{smallmatrix} \right\}_{r+1}}{\binom{n+q+j}{n}}.$$

**Proof.** By setting  $f(x) = B_n^{(x)}(r)$  in (8) we obtain

$$B_n^{(\alpha)}(r) = (\alpha + q) \binom{\alpha + q + p}{p} \sum_{j=0}^p (-1)^j \binom{p}{j} \frac{B_n^{(-j-q)}(r)}{\alpha + q + j}.$$

On using (5), the last identity being the first identity of the proposition.

Upon using the Carlitz's identity  $b_n^{(\alpha)}(x) = B_n^{(n+1-\alpha)}(x+1)$ , see [3, Eqs. (2.11), (2.12)], the second identity is equivalent to the first one.  $\square$

For  $p = n$  and  $q = 0$  in Proposition 3, we get the following corollary.

**Corollary 4** *Let  $\alpha$  be a real number and  $r, n$  be non-negative integers. We have*

$$B_n^{(\alpha)}(r) = \alpha \binom{\alpha + n}{n} \binom{2n}{n}^{-1} \sum_{j=0}^n \frac{(-1)^j}{\alpha + j} \binom{2n}{n+j} \left\{ \begin{smallmatrix} n+r+j \\ r+j \end{smallmatrix} \right\}_r,$$

$$b_n^{(\alpha)}(r) = (n + 1 - \alpha) \binom{2n + 1 - \alpha}{n} \binom{2n}{n}^{-1} \sum_{j=0}^n \frac{(-1)^j}{n + 1 - \alpha + j} \binom{2n}{n+j} \left\{ \begin{smallmatrix} n+r+j+1 \\ r+j+1 \end{smallmatrix} \right\}_{r+1}.$$

In particular, for  $\alpha = 1$  the values of the classical Bernoulli polynomials at non-negative integers are

$$B_n(r) : = B_n^{(1)}(r) = (n + 1) \binom{2n}{n}^{-1} \sum_{j=0}^n \frac{(-1)^j}{j + 1} \binom{2n}{n+j} \left\{ \begin{smallmatrix} n+r+j \\ r+j \end{smallmatrix} \right\}_r,$$

$$b_n(r) : = b_n^{(1)}(r) = n \sum_{j=0}^n \frac{(-1)^j}{n + j} \binom{2n}{n+j} \left\{ \begin{smallmatrix} n+r+j+1 \\ r+j+1 \end{smallmatrix} \right\}_{r+1}.$$

These representations show that the classical Bernoulli numbers admit the representations

$$B_n = (n + 1) \binom{2n}{n}^{-1} \sum_{j=0}^n \frac{(-1)^j}{j + 1} \binom{2n}{n+j} \left\{ \begin{smallmatrix} n+j \\ j \end{smallmatrix} \right\},$$

$$b_n = n \sum_{j=0}^n \frac{(-1)^j}{n + j} \binom{2n}{n+j} \left\{ \begin{smallmatrix} n+j+1 \\ j+1 \end{smallmatrix} \right\}.$$

Note that the above formula of Bernoulli numbers  $B_n$  is exactly the formula given in [10, Thm. 3.1].

**Remark 5** The Genocchi numbers  $(G_n; n \geq 0)$  given by  $G_n = 2(1 - 2^{2n}) B_{2n}$ ,  $n \geq 1$ , see [5, Proposition 2.1], can be written via the above two expressions of  $B_n$  as

$$\begin{aligned} G_n &= 4n(1 - 2^{2n}) \sum_{j=0}^{2n} \frac{(-1)^j}{2n+j} \binom{4n}{2n+j} \begin{bmatrix} 2n+j+1 \\ j+1 \end{bmatrix}, \\ G_n &= 2(2n+1)(1 - 2^{2n}) \binom{4n}{2n}^{-1} \sum_{j=0}^{2n} \frac{(-1)^j}{j+1} \binom{4n}{2n+j} \left\{ \begin{matrix} 2n+j \\ j \end{matrix} \right\}. \end{aligned}$$

Furthermore, the link between the Euler and Bernoulli polynomials via the identity  $E_{n-1}(2x) = \frac{2}{n}(B_n(2x) - 2^n B_n(x))$ , see [16, p. 88], shows that the values of the Euler polynomials at even integers can be written on using the above expressions of  $B_n(-r)$  and  $B_n(r)$  as

$$E_{n-1}(-2r) = \frac{2}{n}(B_n(-2r) - 2^n B_n(-r)) \quad \text{and} \quad E_{n-1}(2r) = \frac{2}{n}(B_n(2r) - 2^n B_n(r)).$$

**Remark 6** It is known that  $B_n = \sum_{j=0}^n (-1)^j \frac{j!}{j+1} \{n \atop j\}$ . Similarly of the proof of this identity, we have

$$\sum_{n \geq 0} B_n(r) \frac{t^n}{n!} = \frac{\ln(1 + \exp(t) - 1)}{\exp(t) - 1} \exp(rt) = \sum_{j \geq 0} (-1)^j \frac{j!}{j+1} \left( \frac{1}{j!} (\exp(t) - 1)^j \exp(rt) \right)$$

which gives

$$B_n(r) = \sum_{j=0}^n (-1)^j \frac{j!}{j+1} \left\{ \begin{matrix} n+r \\ j+r \end{matrix} \right\}_r.$$

### 3 Identities linked $r$ -Stirling numbers and binomial coefficients

The above Propositions can be used to deduce relations between the  $r$ -Stirling numbers and binomial coefficients as it is shown by the following two corollaries..

**Corollary 7** Let  $r, n, p, q, k$  be non-negative integers with  $p \geq n$ . We have

$$\begin{aligned} & \sum_{j=0}^p (-1)^j \binom{j+q}{q} \binom{n+k+q+j}{k} \binom{n+k+p+q+1}{p-j} \begin{bmatrix} n+r+j+q \\ r+j+q \end{bmatrix}_r \\ &= \binom{n+k+q}{q} \sum_{j=0}^n (-1)^{n-j} \binom{n+k}{j+k} \left\{ \begin{matrix} j+k \\ k \end{matrix} \right\} (r-1)^{n-j} \end{aligned}$$

and

$$\begin{aligned} & \sum_{j=0}^p (-1)^j \binom{j+q}{q} \binom{n+k+q+j}{k} \binom{n+k+p+q+1}{p-j} \left\{ \begin{matrix} n+r+j+q \\ r+j+q \end{matrix} \right\}_r \\ &= \binom{n+k+q}{q} \sum_{j=0}^n (-1)^j \binom{n+k}{j+k} \begin{bmatrix} j+k \\ k \end{bmatrix} (r-1)^{n-j}. \end{aligned}$$

**Proof.** For  $\alpha = n + k + 1$  in the first identities of Propositions 1 and 3, the left hand side of the first identity of this corollary is equal to  $\binom{n+k}{k} b_n^{(n+k+1)}(-r)$  and the left hand side of the second identity is equal to  $\binom{n+k}{k} B_n^{(n+k+1)}(r)$  and use the fact that:

$$\begin{aligned}
B_n^{(n+k+1)}(x) &= b_n^{(-k)}(x-1) \\
&= \frac{d^n}{dt^n} \left( \left( \frac{\ln(1+t)}{t} \right)^k (1+t)^{x-1} \right) \Big|_{t=0} \\
&= \frac{n!}{(n+k)!} \frac{d^{n+k}}{dt^{n+k}} \left( (\ln(1+t))^k (1+t)^{x-1} \right) \Big|_{t=0} \\
&= \frac{n!}{(n+k)!} \sum_{j=0}^{n+k} \binom{n+k}{j} \frac{d^j}{dt^j} (\ln(1+t))^k \frac{d^{n+k-j}}{dt^{n+k-j}} (1+t)^{x-1} \Big|_{t=0} \\
&= \frac{1}{\binom{n+k}{k}} \sum_{j=0}^n (-1)^j \binom{n+k}{j+k} \left[ \begin{matrix} j+k \\ k \end{matrix} \right] (x-1)^{n-j}
\end{aligned}$$

and

$$\begin{aligned}
b_n^{(n+k+1)}(x) &= B_n^{(-k)}(x+1) \\
&= \frac{d^n}{dt^n} \left( \left( \frac{\exp(t)-1}{t} \right)^k \exp((x+1)t) \right) \Big|_{t=0} \\
&= \frac{n!}{(n+k)!} \sum_{j=0}^{n+k} \binom{n+k}{j} \frac{d^j}{dt^j} (\exp(t)-1)^k \frac{d^{n+k-j}}{dt^{n+k-j}} (\exp((x+1)t)) \Big|_{t=0} \\
&= \frac{1}{\binom{n+k}{k}} \sum_{j=0}^n \binom{n+k}{j+k} \left\{ \begin{matrix} j+k \\ k \end{matrix} \right\} (x+1)^{n-j}.
\end{aligned}$$

□

Below, we present some particular cases of Corollary 7.

**Example 1** For  $r = 1$  in Corollary 7 we get

$$\begin{aligned}
&\sum_{j=0}^p (-1)^j \binom{j+q}{q} \binom{n+k+q+j}{k} \binom{n+k+p+q+1}{p-j} \left[ \begin{matrix} j+n+q+1 \\ j+q+1 \end{matrix} \right] \\
&= \binom{n+k+q}{q} \left\{ \begin{matrix} n+k \\ k \end{matrix} \right\}, \\
&\sum_{j=0}^p (-1)^{n-j} \binom{j+q}{q} \binom{n+k+q+j}{k} \binom{n+k+p+q+1}{p-j} \left\{ \begin{matrix} j+n+q+1 \\ j+q+1 \end{matrix} \right\} \\
&= \binom{n+k+q}{q} \left[ \begin{matrix} n+k \\ k \end{matrix} \right].
\end{aligned}$$

**Example 2** For  $k = 0$  in Corollary 7 we get

$$\begin{aligned}
\sum_{j=0}^p (-1)^{n-j} \binom{j+q}{q} \binom{n+p+q+1}{p-j} \left[ \begin{matrix} n+r+j+q \\ r+j+q \end{matrix} \right]_r &= \binom{n+q}{q} (r-1)^n, \\
\sum_{j=0}^p (-1)^j \binom{j+q}{q} \binom{n+p+q+1}{p-j} \left\{ \begin{matrix} n+r+j+q \\ r+j+q \end{matrix} \right\}_r &= \binom{n+q}{q} (r-1)^{\underline{n}}.
\end{aligned}$$

**Example 3** For  $n = 0$  in Corollary 7 we get

$$\sum_{j=0}^p (-1)^j \binom{j+q}{q} \binom{k+q+j}{k} \binom{k+q+p+1}{p-j} = \binom{k+q}{q}.$$

**Corollary 8** Let  $r, n, p, q, k$  be non-negative integers with  $p \geq n$ . We have

$$\begin{aligned} & \sum_{j=0}^p (-1)^j \binom{n+p+q+k+1}{p-j} \binom{q+j}{j} \binom{q+k+1+j}{k} \left[ \begin{matrix} n+r+q+k+1+j \\ r+q+k+1+j \end{matrix} \right]_r \\ &= (-1)^n \binom{n+p+q+k+1}{n+k} \left[ \begin{matrix} n+k+r \\ k+r \end{matrix} \right]_r \end{aligned}$$

and

$$\begin{aligned} & \sum_{j=0}^p (-1)^j \binom{n+p+q+k+1}{p-j} \binom{q+j}{j} \binom{q+k+1+j}{k} \left\{ \begin{matrix} n+r+q+k+1+j \\ r+q+k+1+j \end{matrix} \right\}_r \\ &= \binom{n+p+q+k+1}{n+k} \left\{ \begin{matrix} n+k+r \\ k+r \end{matrix} \right\}_r. \end{aligned}$$

**Proof.** Choice  $\alpha = -k$  with  $k+1 \leq q$  in the first identities of Propositions 1 and 3, the left hand side of the first identity of this corollary is equal to  $\binom{n+k}{k} \binom{n+p+q}{n+k} b_n^{(-k)}(-r)$  and the left hand side of the second identity is equal to  $\binom{n+k}{k} \binom{n+p+q}{n+k} B_n^{(-k)}(r)$  and use the identities given by (5) and (6). After that, replace  $q$  by  $q+k+1$ .  $\square$

**Example 4** For  $k = 0$  in Corollary 8 we get

$$\begin{aligned} & \sum_{j=0}^p (-1)^j \binom{n+p+q+1}{p-j} \binom{q+j}{j} \left[ \begin{matrix} n+r+q+1+j \\ r+q+1+j \end{matrix} \right]_r = \binom{n+p+q+1}{n} r^{\bar{n}}, \\ & \sum_{j=0}^p (-1)^j \binom{n+p+q+1}{p-j} \binom{q+j}{j} \left\{ \begin{matrix} n+r+q+1+j \\ r+q+1+j \end{matrix} \right\}_r = \binom{n+p+q+1}{n} r^n. \end{aligned}$$

**Example 5** For  $n = 0$  in Corollary 8 we get

$$\sum_{j=0}^p (-1)^j \binom{p+q+k+1}{p-j} \binom{q+j}{q} \binom{q+k+1+j}{k} = \binom{p+q+k+1}{k}.$$

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